Introduction

Bose-Einstein condensation is basically a manifestation of macroscopic occupation of a single quantum state. The idea of such a macroscopic occupation dates back to 1924–1925, when Albert Einstein extended the statistical arguments presented by Satyendra Nath Bose to systems consisting of a conserved number of bosonic particles. Einstein recognised that at sufficiently low temperatures the quantum statistical distribution of an ideal gas of bosons shows condensation of a macroscopic fraction of the material into the ground state of the system. This phenomenon, subsequently termed Bose–Einstein condensation (BEC), is a unique, purely quantum mechanical phase transition in the sense that it occurs in principle even in noninteracting bosonic systems. Nowadays BECs are routinely produced in research laboratories around the world and they provide a unique opportunity to study fundamental quantum phenomena.

Figure 1. Visualization of the first quantum knot. (Credit: David Hall)
Recently, we created and observe knot-like structures referred to as knot solitons in quantum-mechanical order parameter describing a BEC. See the attached manuscript that was published in Nature Physics. Although knots have been tied in the classical ropes for millennia and considered in classical fields for more than a century, no one had previously observed a single knot in the context of quantum dynamics.

**Project goals**

Currently, the knots are created in the polar order parameter of a $^{87}$Rb condensate which tends to decay into the ferromagnetic phase. This decay will also destroy the knots. Your project is to study the dynamics of the created knots and find ways for their stabilization in BECs.

**Research site**

Your site of research will be the premises of Quantum Computing and Devices, the so-called QCD Labs, on the Otaniemi campus of Aalto University. There are both theorists and experimentalists working in the group. See the group website (http://physics.aalto.fi/groups/comp/qcd/) for more information.

**Instructors**

Your instructor will be M.Sc. Tuomas Ollikainen supported by Dr. Mikko Möttönen.

**Working methods**

Your work will involve building the theoretical understanding of BECs and monopoles. You will also carry out numerical modeling using CUDA.

**Thesis possibilities**

Depending on your level, this project can be adjusted for a BSc thesis, special assignment, MSc thesis, or a PhD thesis project. Prior knowledge of quantum mechanics and an excellent study record is a prerequisite.
Tying Quantum Knots*

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Knots are familiar entities that appear at a captivating nexus of art, technology, mathematics, and science [1]. As topologically stable objects within field theories, they have been speculatively proposed as explanations for diverse persistent phenomena, from atoms and molecules [2] to ball lightning [3] and cosmic textures in the universe [4]. Recent experiments have observed knots in a variety of classical contexts, including nematic liquid crystals [5–7], DNA [8], optical beams [9, 10], and water [11]. However, no experimental observations of knots have yet been reported in quantum matter. We demonstrate here the controlled creation [12] and detection of knot solitons [13, 14] in the order parameter of a spinor Bose–Einstein condensate. The experimentally obtained images of the superfluid directly reveal the circular shape of the soliton core and its accompanying linked rings. Importantly, the observed texture corresponds to a topologically non-trivial element of the third homotopy group [15] and demonstrates the celebrated Hopf fibration [16], which unites the closed curve to the existence and dynamics of knotted vortex rings in an ethereal fluid [2].

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Knots can also appear as particle-like solitons in classical and quantum fields [21], the nature of which has been a subject of intense mathematical interest for more than eighty years [16]. In this case, the closed curve is a ring that describes the core of the soliton. This ring is surrounded by an infinite number of similar rings, each linked with all of the others to generate a knotted field structure [22]. For example, the classical Maxwell’s equations admit solutions that involve knot solitons in which the rings are electric and magnetic field lines [23]. Quantum-mechanical examples have been theoretically proposed [13, 14] for the Faddeev–Skyrme model, in which each of the linked rings consists of the points in space sharing a particular direction of the field.

In general, knot solitons are non-singular topological excitations [15] that change smoothly and non-trivially in all three spatial dimensions. They are therefore described by the third homotopy group, \( \pi_3 \), which classifies such textures according to whether or not they can be continuously transformed into one another. One-dimensional solitons and singular vortex lines, both belonging to the fundamental group \( \pi_1 \), have been identified experimentally in superfluids [24–26], as have two-dimensional skyrmions [27] and singular monopoles [28] belonging to the second homotopy group, \( \pi_2 \). Since singular defects within \( \pi_3 \) are unrealisable monopoles in four dimensions, these knotted solitons necessarily belong to the third homotopy group.

Figure 1 | Structure of the knot soliton and the method of its creation. a,b, Schematic magnetic field lines before (a) and during (b) the knot formation, with respect to the condensate (green ellipse). c, Knot soliton configuration in real space and its relation to the nematic vector \( \hat{d} \) in \( S^2 \) (inset). The inner white ring \( (d_z = -1, m = 0) \) is the core of the knot soliton. The surrounding coloured bands \( (d_z = 0, m = \pm 1) \) define the surface of a torus, with colours representing the azimuthal angle of \( \hat{d} \) which winds by \( 2\pi \) in both the toroidal and poloidal directions. The outer dark grey rings \( (d_z = 1, m = 0) \) indicate the boundary of the soliton. d,e, When tying the knot, the initially z-pointing nematic vector (black arrows) precesses about the direction of the local magnetic field (cyan lines) to achieve the final configuration (coloured arrows). The dashed grey line shows where \( d_z = 0 \), the white line indicates the soliton core \( (d_z = -1) \), and the dark grey line defines the boundary of the volume \( (d_z = 1) \).
The polar order parameter may therefore be expressed in terms of the nematic vector, \( \hat{d} \), defined by equation (2). The nematic vector field \( \hat{d}(r) \) maps points in real space \( r \in \mathbb{R}^3 \) to points on the surface of the unit sphere \( \hat{d} \in S^2 \).

In our case, the nematic vector assumes a constant value, \( \hat{d} = 1 \), representing a zero gap soliton.

In this Letter we demonstrate the creation and observation of knot solitons in the polar phase of a spinor Bose–Einstein condensate. We adopt the theoretical method proposed in ref. [12] and implement it using experimental techniques that have recently been used to create Dirac monopoles [29] and isolated monopoles [28]. An overview of the experiment is given in Fig. 1. The core of the knot soliton is first observed as a ring of enhanced particle density at the periphery of the condensate, which is obtained by spin rotations

\[
\mathcal{D}(\alpha, \beta) \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -\sqrt{2} \cos \beta \\ -e^{i\alpha} \sin \beta \\ e^{i\alpha} \sin \beta \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -d_x + id_y \\ \sqrt{2}d_z \\ d_x + id_y \end{array} \right)
\]

(2)

for angles \( \beta \) and \( \alpha \) about the \( y \) and \( z \) axes, respectively. The polar order parameter may therefore be expressed as

\[
\Psi(r, t) = \sqrt{n(r, t)}e^{i\phi(r, t)}\zeta(r, t)
\]

(1)

where \( n \) is the atomic density, \( \phi \) is a scalar phase, and \( \zeta = (\zeta_1, \zeta_0, \zeta_{-1})^T \) is a three-component \( z \)-quantized spinor

with \( \zeta_m = \langle m | \zeta \rangle \). We restrict our attention here to the polar phase of the condensate, which is obtained by spin rotations

\[
\mathcal{D}(\alpha, \beta) \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -\sqrt{2} \cos \beta \\ -e^{i\alpha} \sin \beta \\ e^{i\alpha} \sin \beta \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -d_x + id_y \\ \sqrt{2}d_z \\ d_x + id_y \end{array} \right)
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for angles \( \beta \) and \( \alpha \) about the \( y \) and \( z \) axes, respectively. The polar order parameter may therefore be expressed as

\[
\Psi(r, t) = \sqrt{n(r, t)}e^{i\phi(r, t)}\hat{d}(r)
\]

(3)

in terms of the nematic vector, \( \hat{d} \), defined by equation (2).

The nematic vector field \( \hat{d}(r) \) maps points in real space \( r \in \mathbb{R}^3 \) to points on the surface of the unit sphere \( \hat{d} \in S^2 \). In our case, the nematic vector assumes a constant value,
In this experiment we consider the Hopf map [16], which has $Q = 1$ and is generated physically in our system by spin rotations in an inhomogeneous magnetic field. We begin with an optically trapped $^{87}$Rb condensate (see Methods) described by the spinor $\xi = (0, 1, 0)^T$, corresponding to $\mathbf{d} = \mathbf{d}_0 = \hat{z}$. The inhomogeneous magnetic field is given by

$$\mathbf{B}(\mathbf{r}', t) = b_q(x'\hat{x} + y'\hat{y} - z'\hat{z}) + \mathbf{B}_b(t)$$

where the condensate is taken to be at the origin of the rescaled coordinate system $x' = x$, $y' = y$, and $z' = 2z$. In the gradient $b_q = 4.5$ G/cm and effective bias field $B_z \sim 30$ mG, the zero point of the magnetic field is initially $33 \mu$m away from the centre of the condensate. The creation of the knot begins with a sudden change of $\mathbf{B}_b(t)$ that places the field zero at the centre of the condensate, ideally leaving its state unchanged (see Fig. 1a,b and Extended Data Fig. 1). The nematic vectors then precess about the direction of the local magnetic field at their spatially-dependent Larmor frequencies

$$\omega_L(\mathbf{r}') = \frac{g_F B_b |\mathbf{B}(\mathbf{r}')|}{\hbar} = \frac{g_F B_b b_q}{\hbar}$$

where $g_F$ is the atomic Landé $g$-factor, $\mu_B$ is the Bohr magneton, and $r' = \sqrt{x'^2 + y'^2 + z'^2}$. The optimal result is the time-dependent nematic vector field

$$\hat{d}(\mathbf{r}') = \exp \left[ -i\omega_L(\mathbf{r}') t \right] \hat{B}(\mathbf{r}') \cdot \mathbf{F} \hat{d}_0$$

where $\mathbf{F}$ is the vector of dimensionless spin-1 matrices in the Cartesian basis. Importantly, $\hat{d}(\mathbf{r}') = \hat{d}_0$ for all points satisfying $\omega_L(\mathbf{r}') t = 2\pi$, and hence we may choose the volume $V$ to be a ball of radius

$$R' = \frac{2\pi \hbar}{g_F \mu_B b_q}.$$

Figure 1d–e illustrates how the nematic vector assumes its knot soliton configuration as a result of the spatially dependent Larmor precession. The core of the soliton is identified with the preimage of the south pole of $S^2$, i.e., $\mathbf{d}_{\text{core}} = -\mathbf{d}_0$, which lies in the $x' y'$-plane. This ring defines a circle (Fig. 1c–e) in which the condensate is entirely in the $m = 0$ spinor component [equation (2)]. The comparable preimage of the north pole, $\hat{\mathbf{d}} = \mathbf{d}_0$, includes the $z'$-axis and all points on the boundary of $V$. Because antipodal points on $S^2$ correspond to the same spinor up to a sign [equation (3)], this preimage is also entirely in the $m = 0$ component. The preimages of the equatorial points on the two-sphere consist of linked rings that define a toroidal tube enclosing the core, as shown in Fig. 1c. Since $d_z = 0$ at the equator of $S^2$, this torus consists of overlapping $m = \pm 1$ components [equation (2)]. Elsewhere, $\hat{\mathbf{d}}$ smoothly interpolates between these values.

After an evolution time $T_{\text{evolve}}$ we apply a projection ramp in which the bias field $B_z$ is rapidly changed to $\hat{d}_0$, at the boundary of a certain volume $V$. We restrict our studies to textures inside the volume $V$, which as a result can be identified with $S^1$, the surface of a four-dimensional ball. Nontrivial mappings $\hat{\mathbf{d}}(\mathbf{r})$ from $S^3$ to $S^2$ lead to knotted field configurations characterised by integer topological charges or Hopf invariants [21], $Q$, as determined by the third homotopy group $\pi_3(S^2) = \mathbb{Z}$. Field configurations with different Hopf invariants cannot be continuously deformed into one another and are therefore topologically distinct.

Taken together, the points in $V$ at which $\hat{\mathbf{d}}$ assumes the same value, $\hat{\mathbf{d}}_q$, define a closed curve known as the preimage of $\mathbf{d}_q$. Each of these preimages is linked with all of the others, which are associated with different $\hat{\mathbf{d}}$, exactly $Q$ times. Thus the linking number is equivalent to the Hopf invariant [21], and provides an alternative perspective on its physical significance.
move the field zero far from the centre of the condensate [28, 29]. The condensate is then released from the optical trap, whereupon it expands and falls under the influence of gravity. Subsequently, its spinor components are separated and imaged simultaneously along both the vertical (z) and horizontal (y) axes. The temporal evolution of the particle column densities in the \( m = 0 \) component, \( \int nd^2 y \), is shown in Fig. 2a,b. The pictures show the combined preimages of the poles of \( S^2 \) (\( d_z = \pm 1 \)), revealing in one picture both the core of the knot and the boundary of \( V \). The preceding analytical result for the radius of the core, \( R'/2 \) from equation (7), agrees well with the experimental observations.

Figure 3 provides a detailed comparison of the experimentally obtained knot soliton with numerical simulations of the corresponding Gross–Pitaevskii equation (see Methods) with no free parameters. The very good correspondence between the experiment and the simulation, together with the qualitatively correct behaviour of the \( m = \pm 1 \) spinor components that jointly accumulate in the vicinity of the intensity minima of the \( m = 0 \) component, provide further evidence that the observed texture is that of a knot soliton. Note that the \( m = \pm 1 \) components do not overlap as a result of the time-of-flight imaging technique (see Extended Data Fig. 2 for simulated images of the spinor components prior to expansion).

By definition the nematic vector is aligned with the local spin quantization axis, along which the condensate is fully in the \( m = 0 \) component [see equation (2)]. Remarkably, a projection ramp taken along an arbitrary axis, \( \eta \), populates the \( m = 0 \) component with the preimages of the antipodal points in \( S^2 \) corresponding to \( d_\eta = \pm 1 \). By performing the projection ramp along \( x \) and \( y \), for example, we can observe the linked preimages of \( d_x = \pm 1 \) and \( d_y = \pm 1 \), respectively, in the \( m = 0 \) component (Fig. 4). These images explicitly demonstrate the linked rings of the Hopf fibration and provide conclusive evidence of the existence of the knot soliton.

Our observations suggest future experiments on the dynamics, stability, and interactions of knot solitons [19]. Experimental creation of multiply-charged and knotted-core solitons in quantum fields stands as another promising research direction. Furthermore, stabilising the knot soliton against dissipation, a feature associated with textures in the Faddeev–Skyrme model [13, 14], remains an important experimental milestone.

**METHODS**

**Experiment.** The condensate initialisation, trapping, and imaging techniques are essentially identical to those of ref. 28. The key technical difference in the present experiments is that we bring the magnetic field zero suddenly into the condensate centre, in contrast to the adiabatic creation ramp in ref. 28. Extended Data Fig. 1 shows the measured temporal evolution of the electric current controlling \( B_z \) during its excursion, expressed in units of the magnetic field. We define \( t = 0 \) to be the moment at which the field zero has traversed 90% of the distance towards its final location at the centre of the condensate. The strength of the quadrupole gradient field is estimated by repeating the knot creation experiment with a small bias field offset applied along the \( x \)-axis, which introduces a fringe pattern that winds at a rate proportional to the strength of the gradient.

The crossed-beam optical dipole trap operates at 1064 nm with frequencies \( \omega_r \sim 2\pi \times 130 \) Hz and \( \omega_z \sim 2\pi \times 170 \) Hz in the radial and axial directions, respectively. The total number of particles in the condensate at the moment of imaging is typically \( 2.5 \times 10^5 \).

**Data.** The experimentally obtained images of knot solitons shown in this manuscript represent typical results selected from among several hundred successful realisations taken under similar conditions over the course of more than a year. Remarkably, almost identical knot solitons have been created with several minutes of time elapsed between the realisations while not changing the applied control sequences.

**Simulation.** We theoretically describe the low-temperature dynamics of the condensate using the full three-dimensional spin-1 Gross–Pitaevskii equation

\[
i\hbar \partial_t \Psi(r) = \{ \hbar(r) + n(r)\}c_0 + c_2 S(r) \cdot \mathbf{F} - i\Gamma r^2(r)\} \Psi(r)
\]

where we denote the single-particle Hamiltonian by \( h(r) \), the spin vector by \( S(r) = \zeta(r)\mathbf{F} \zeta(r) \), and the density–density and spin–spin coupling constants by \( c_0 = 4\hbar^2(a_0 + 2a_2)/(3m) \) and \( c_2 = 2\hbar^2(a_2 - a_0)/(3m) \), respectively. We employ the literature values for the three-body recombination rate \( \Gamma = 2.9 \times 10^{-30} \) cm\(^3\)/s, the \( ^{87}\text{Rb} \) mass \( m = 1.443 \times 10^{-25} \) kg, and the s-wave scattering lengths \( a_0 = 5.387 \) nm and \( a_2 = 5.313 \) nm. The single-particle Hamiltonian assumes the form \( h(r) = -\hbar^2 \nabla^2/(2m) + V_{\text{opt}}(r) + g_F \mu_B \mathbf{B}(r, t) \cdot \mathbf{F} + q |\mathbf{B}(r, t) \cdot \mathbf{F}|^2 \) where the strength of the quadratic Zeeman effect is given by \( q = 2\pi \hbar \times 2.78 \) MHz/T and the optical trapping potential is approximated by \( V_{\text{opt}}(r) = m\omega_z^2(x^2 + y^2) + m\omega_y^2 z^2 \). The Gross–Pitaevskii equation is integrated using a split-operator method and fast Fourier transforms on a discrete grid of size \( 8 \times 10^6 \). The computations are carried out using state-of-the-art graphics processing units. The simulations reproduce the experimental results with no free parameters: Only literature values for constants and independently measured parameters, such as the temporal dependence of the magnetic field, are employed. The magnetic field gradient that is briefly applied to separate the different spinor components during the time-of-flight imaging is not included in the simulations.


Extended Data Figure 1 | The rapid change to the applied magnetic field. Typical averaged trace of the change in the applied current to the magnetic field coils that initiates the knot creation process. The vertical axis is expressed in terms of $B_z$ using coil calibration data obtained from microwave spectroscopy. Some field contributions from external sources, such as eddy currents, are not included. The grey region shows the effective extent of the condensate as determined by the value of the magnetic field gradient.

Extended Data Figure 2 | Numerical simulation of the knot creation before expansion. Horizontally (a) and vertically (b) integrated particle densities of a condensate just before the projection ramp after an evolution time of 557 $\mu$s, with matching parameters as in Fig. 3. The field of view is 15 $\mu$m × 15 $\mu$m in each frame, and the maximum pixel intensity corresponds to $\tilde{n}_p = 3.8 \times 10^{11}$ cm$^{-2}$. 